Emulation of Stationary Moving Medium by Magneto-Electric Material in the Finite Element Method

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The numerical simulation of electromagnetic phenomena involving moving bodies is always challenging when the effect of motion becomes significant. We propose a novel method by which the first order "motional term" of the partial differential equation (PDE) can be incorporated into PDE coefficients normally describing material characteristics. The method is general in the sense that various types of moving media (e.g. metal, dielectric) can be treated in a similar way. The most straightforwardly it can be utilized in the analysis of models with stationary geometry, and it may have some advantages over other approaches in this field. The method is demonstrated through examples.

Index Terms—**Material modeling, magneto-electric effect, moving medium**

I. Introduction

Electromagnetic problems involving moving bodies have to be analyzed frequently in the practice. In certain cases the moving domain can be taken unbounded in the direction of motion and its cross section invariant to the movement (translation or rotation), in other words, the geometry is stationary. In this manner can be modeled among others electromagnetic launchers [1], the homopolar generator (or Faraday disc) and the rather "exotic" Wilson&Wilson experiment [2].

It is characteristic to all these problems that the governing second order partial differential equation (PDE) —derived from Maxwell's equations— is extended by a first order term regarding the moving medium. In some finite element method (FEM) programs (e.g. in Comsol Multiphysics® [3]) an induced Lorentz current density term for the moving conducting medium is implemented, as it is deemed the most important effect of motion. Nevertheless a generic PDE template of this kind —e.g. for treating moving dielectrics— is not provided out-of-the-box in the available FEM programs, as a rule. In the lack of an appropriate predefined "problem type", the user who is bound to a particular software commonly thinks in terms of the following two solutions:

a) If the PDE parameters can be specified flexibly in the program, i.e. arbitrary expressions containing coordinates and field values can be set, then —as a workaround— the first order term may be put to the right hand side as part of the excitation. However, such term acts *formally* as a kind of "nonlinear constraint", and may require nonlinear solver.
b) In case the user has sufficient access to (and skill of) controlling the software at a lower level, the weak form of the corresponding PDE can be implemented (see e.g. [4]). Note however, that stability and convergence problems might occur due to the first order term when the motional effect is significant, and that sometimes special techniques like *upwinding* must be used to overcome this [5].

The original motivation of this work is to implement and solve problems involving stationary moving media within a general purpose FEM environment, *without* resorting to the above mentioned workarounds. The proposed method is based on the conversion of movement to specific material parameters. When doing this we are aware of the fact that most of the available FEM softwares are capable of treating inhomogeneous and anisotropic media. Besides the formal elegance of this method, the drawbacks of solutions a) and b) are partly avoided: a linear solver can be used, and also the convergence properties may be improved, respectively.

II. Theory

The moving medium can generate magneto-electric coupling, because the Lorentz-transformation from the co-moving frame to the laboratory frame naturally mixes electric and magnetic fields. This phenomenon is reflected well in Ohm's law and Minkowski's constitutive relations:

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{1}$$

$$\boldsymbol{D} = \varepsilon_0\varepsilon_r\boldsymbol{E} + \frac{\varepsilon_r\mu_r - 1}{c_0^2}\boldsymbol{v} \times \boldsymbol{H} \tag{2}$$

$$\boldsymbol{B} = \mu_0\mu_r\boldsymbol{H} - \frac{\varepsilon_r\mu_r - 1}{c_0^2}\boldsymbol{v} \times \boldsymbol{E} \tag{3}$$

These equations apply to moving linear isotropic media and represent a quasi-relativistic first order approximation (i.e. terms of order v^2/c^2 are neglected) which is valid for $v \ll c$ (see e.g. [6]). The symbols $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{D}$ and $\boldsymbol{J}$ stand for electric field, magnetic flux density, magnetic field, electric displacement and current density, respectively, and $\boldsymbol{v}$ is the velocity of the medium, all these vector fields being measured in the laboratory frame. As opposed to this, the relative permittivity and permeability of the medium, ε_r and μ_r, and specific conductivity σ are measured naturally in the co-moving frame. Finally, ε_0, μ_0 and c_0 are the permittivity, permeability and speed of light in vacuum.

Similar magneto-electric coupling can be generated by specific composite meta-materials (at rest) too. Actually, this property is long since known and exploited in transformation optics (TO) among others to raise the illusion of motion [7]. It is all the more surprising therefore that this relationship was

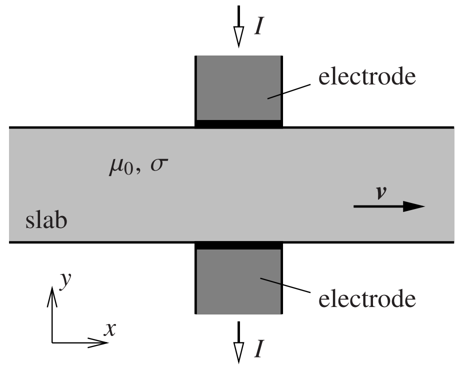

Fig. 1. The studied problem: "Sliding contacts"

not exploited in the opposite way, i.e. for modeling real motion with fictitious material parameters; only very recently we can find a somewhat similar 1D approach [8].

III. Illustrative example

The method is demonstrated here on a simple example of sliding contacts (see [6] p.137). A conducting non-magnetic slab is moving between two electrodes with constant velocity, while a fixed DC current, I, is injected through it (Fig. 1). Since the current distribution in the slab is stationary from the point of view of the electrodes, it is worth to do the modeling in the rest frame of the latter.

Following the reasoning of [6] one can see that the in-plane current distribution can be expressed with the perpendicular component of the magnetic field as $\boldsymbol{J} = \nabla H_z \times \boldsymbol{e}_z$, with $\boldsymbol{e}_z$ being the unit vector in the z direction, and that for H_z the following scalar PDE can be written:

$$\nabla^2 H_z - \mu_0 \sigma \boldsymbol{v} \cdot \nabla H_z = 0. \tag{4}$$

We examine here an aluminum slab of thickness 10 mm, dragged with velocity 6 m/s in the x direction. The injected current is prescribed by appropriate boundary conditions (further details are omitted herein due to space constraints).

We intend to use the PDE Toolbox of Matlab®, a simplistic but quite flexible 2D FEM environment [9]. There is no predefined PDE that matches (4) in it, so we fall back to the generic elliptic PDE template,

$$-\nabla \cdot (c \nabla u) + au = f, \tag{5}$$

in which u is the unknown scalar field and c, a, f are parameters, which can be functions of the location and even of the field itself.

A. Using the nonlinear solver of PDE Toolbox

For obtaining a reference solution, first we follow the way described in point a) of section I. Considering $u \equiv H_z$ we set the parameters of (5) according to (4) as

$$c = 1, \qquad a = 0, \qquad f = -\mu_0 \sigma v_x \partial_x u \tag{6}$$

(note that partial derivatives of the scalar field, $\partial_x u$ and $\partial_y u$, are available in Matlab under the names 'ux' and 'uy'). Because of the u-dependent expression of f we are forced to use the nonlinear solver, which makes 5 iterations in 2.58 seconds. The streamlines of $\boldsymbol{J}$ are plotted in Fig. 2, in which the effect of motion is apparent.

B. Substituting inhomogeneous anisotropic material

We developed a straightforward method (detailed in the full paper) by which the first order motional term can be assimilated into artificial material parameters. In the studied problem we can use the following settings:

$$\bar{\bar{c}} = \begin{pmatrix} 1 & -\mu_0 \sigma v_x y \\ \mu_0 \sigma v_x y & 1 \end{pmatrix}, \quad a = 0, \quad f = 0 \tag{7}$$

(we utilize that c can be a tensor in Matlab). In this case we can use the linear solver, which computes the field in one step taking only 0.48 seconds. The current distribution (not shown) is practically indistinguishable from that of the previous solution. Remarkably, although the material parameters in (7) seem to depend on the choice of the origin (y=0), the computed field does not, of course.

The proposed method is rather general. For instance, we successfully solved the problem "Scattering by a rotating circular dielectric cylinder" (see [6] p.298) with it. Also the Wilson&Wilson experiment [2] can be modeled using the same apparatus. The full paper will contain —in addition to the details of the method— a more complex 3D example, as well as a study on the convergence properties.

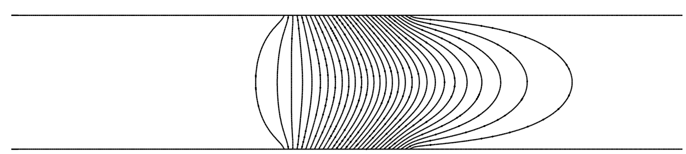

Fig. 2. Streamline plot of the current density distribution in the slab

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